

ISLAMIC TILINGS OF THE ALHAMBRA PALACE: TEACHING THE BEAUTY OF MATHEMATICS



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Introduction

How were the ancient Greeks able to measure the circumference of the earth? How did the Babylonians discover the Pythagorean Theorem two thousand years before Pythagoras? How were 13th Century Islamic mathematicians able to develop mathematical models that would call into dispute the longstanding theory on planetary motion laid down centuries earlier by Ptolemy? What



Figure 1 The Alhambra Palace in Granada, Spain

geometry was needed by artisans in the Middle Ages to create the beautiful symmetric tilings of the Alhambra Palace (Figure 1) in Granada, Spain? These open questions are not easy to answer but they are precisely the type of questions that can spark dynamic discussions with students in mathematics classrooms.

Mathematics as Cultural Heritage

In recent years, mathematics educators have investigated teaching methodologies for introducing mathematics with the aid of cultural connections. The NCTM, National Council of Teachers of Mathematics, has as one of its overall goals for students to view

mathematics as a part of cultural heritage. The history of mathematics is a rich tapestry woven of contributions from individuals and groups from throughout the ancient and modern worlds. Mathematical ideas that we now take for granted like the ten Indo-Arabic digits, 0 through 9, may be told with a story that spans two-thousand years and includes chapters from the Indian subcontinent through the Islamic world of the Middle East and ultimately to Europe and the West with several twists and turns along the way.

Exposing students to cultural and historical connections of mathematics may well heighten the global perspective that mathematics is indeed a creative endeavor of the human mind. In addition, if some of the historical connections are made with the specific cultural heritage of the students, then a sense of excitement and personal ownership may result. In my own experience of teaching and lecturing in various countries, when cultural connections are made with the audience, the mathematics becomes enlivened and more stimulating. Whether speaking to students in Chennai about the Indian mathematician Srinivasa Ramanujan or to students in the United Arab Emirates

about the Islamic mathematician and astronomer Nasir al-Din al-Tusi, students get energized and become more involved if the connection is made to their own culture much as one is excited and intrigued by ones own geneology.

One final positive outcome of teaching with cultural perspective takes place on the instructor’s side. With students given the freedom to interpret the historical development of a particular mathematical idea, they may have an opinion that runs contrary to the norm. Having students voice new ways of looking at an old mathematical idea may cause the instructor to take a fresh look. This “turning the tables” where the student becomes the teacher and the teacher becomes the student can provide rewarding experiences for all involved.

Islamic Culture and Mathematics

For mathematics classes being taught in the Middle East, there are two major areas for consideration in implementing mathematical connections to the Islamic culture. First, the golden age of scientific discovery in the Islamic world from the 9th Century to the 15th Century A.D. provides numerous stories of Islamic mathematicians (Figure 2) and astronomers who contributed to the modern theories of mathematics that are used today. Second, the highly symmetric Islamic art and ornamentation found in mosques, mausoleums, and minarets throughout the world provide interesting examples of geometric construction for use in the mathematics classroom. For this investigation, we focus on the geometric ornamentation of the Alhambra Palace in Granada, Spain and the mathematical concepts used in their construction with an eye towards implementing these ideas in courses ranging from geometry and abstract algebra to general education.

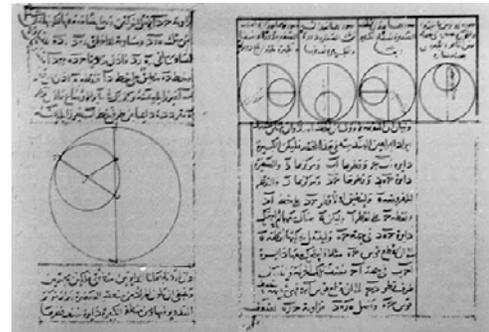


Figure 2 Nasir al-Din al-Tusi, 13th Century Manuscript on Planetary Motion

The Alhambra Palace, Granada, Spain

This past summer while attending a conference on mathematical symmetry in Granada, Spain, I had the wonderful opportunity to tour the Alhambra Palace perhaps the best-known example of Islamic art and architecture in the West. Built by the Moors beginning in the 13th Century, A.D., on a hill overlooking the old city of Granada, the Alhambra consisted of a palace surrounded by several towers and was designed as a fort. The most likely source the name Alhambra is from the Arabic word alhamra meaning red soil, which refers to the color the earth on which the Alhambra is built. Set against the majestic background of the snow-capped Sierra Nevada Mountains, the Alhambra Palace, along with the patios and fountains of the nearby Generalife gardens, strike an imposing presence in the serene Spanish countryside.



Figure 3 M.C. Escher’s “Symmetry Drawing E104” © 2003 Cordon Art B.V. – Baarn – Holland. All rights reserved.

Throughout the years, many visitors from the West have been influenced by pilgrimages to the Alhambra. In the 19th Century, the American author, Washington Irving, spent time at the Alhambra writing his book, *Tales of the Alhambra*. This romantic work was instrumental in rediscovering the beauty of the Alhambra for new generations of visitors. In the 1920’s and 1930’s, the Dutch artist M.C. Escher (Figure 3) made two visits to the Alhambra to sketch and catalogue the

ornate patterns that are found in the tiling throughout the palace and surrounding courtyards. It is this “geometry of the Alhambra” which may be used to enhance student interest and understanding of a variety of ideas related to mathematical symmetry.

Islamic Tilings of the Alhambra

The Islamic tile patterns that are found in the Alhambra are governed by rules of geometric repetition, symmetry, and continuity. According to this strict adherence to conformity, a relatively simple motif may be used to produce an increasing pattern, which may continue on to infinity. As a philosophical metaphor, one could say that each individual makes an important contribution to the universe but, in the universal scheme of life, no one individual is any more important than any other. Understanding the mathematics behind the symmetry and construction of these Islamic patterns is the key to then using these intricate patterns as motivational tools in the classroom. It is this interdisciplinary bridge between mathematics and art that allows students to view mathematics as a part of a bigger picture, which includes aesthetics, culture, and history.

The Mathematics of Islamic Tilings – Symmetry and Geometry

The intricate ornamental beauty of the Alhambra tilings (Figure 4) is useful in developing two broad areas of mathematical understanding. First, in understanding the intrinsic symmetry of a pattern, a student may be introduced to the elementary concepts of group theory. This is a useful endeavor for students regardless of whether or not they plan to take up a more rigorous study of abstract algebra. The second and perhaps more encompassing area is in directing students to discover the geometric methods needed to construct the tilings. This has the potential of introducing students to traditional techniques of straightedge and compass constructions as well as modern geometric design with computer software like Geometer’s Sketchpad. By comparing the traditional with the modern, each student may be challenged to use their individual creativity to design their own tilings. Further, if students are exposed to both the symmetry and geometry of tilings, they may begin to understand some of the links between the worlds of algebra and geometry and ideally begin to see mathematics as one complete world as opposed to smaller disjoint ones.

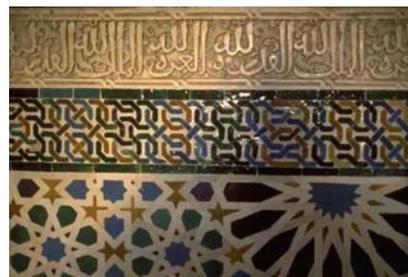


Figure 4 Islamic Tiling from the Alhambra Palace

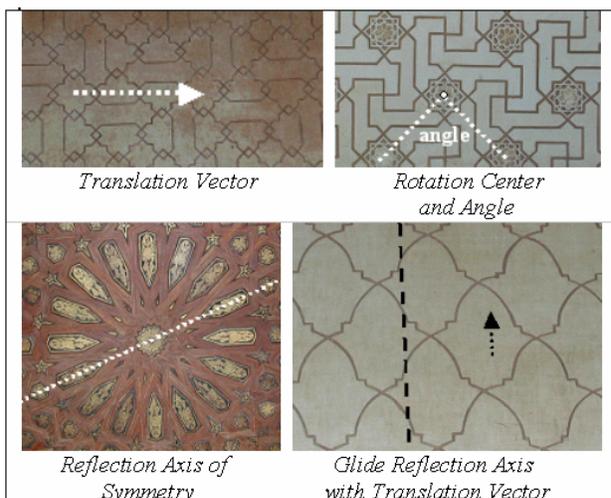


Figure 5 Islamic Tilings from the Alhambra - The Four Types of Symmetries for Patterns in the Plane

are exactly four types of symmetries (Figure 5).

Symmetry

In his 1952 classic book, *Symmetry*, noted mathematician Hermann Weyl began by describing symmetry as the harmony of proportions. The repetitive features found in the tilings of the Alhambra certainly fit Weyl’s definition and to begin to understand the mathematics of these tilings, one must determine a method for measuring this symmetry. To accomplish this, we need a mathematical definition for the symmetry of a pattern.

Definition – A symmetry of a pattern is a rigid motion of a pattern which does not change the appearance of the pattern.

For two-dimensional patterns in a plane, there

Translation – The pattern moves a fixed distance in a linear direction.

Rotation – The pattern is rotated through a given angle about a given center.

Reflection – The pattern moves as a mirror image across a given line.

Glide Reflection – The pattern is translated in a given direction and then reflected in a given line that is parallel to the translation direction.

The collection of symmetries for a particular pattern is called the symmetry group of the pattern. These symmetry groups measure the degree to which a pattern is symmetric or asymmetric and it is possible to have different looking patterns, which have the same symmetry group. In this case, we say that the patterns are isomorphic.



Figure 6 Radial Tiling from the Alhambra

Radial Patterns – Finite Symmetry Groups

Armed with these basic notions of symmetry and symmetry group,



Figure 7 Periodic Tiling from the Alhambra

we can begin to describe the Islamic tilings from the Alhambra in a mathematical manner. We begin by noting that these tiling patterns fall into two major categories. The first type describes patterns, which begin at a central point and grow radially outward (Figure 6). Since these patterns have a central point, they will not have any translations or glide reflections but, may have rotations about the central point or reflections in lines through the central point with the total number of symmetries being finite. These symmetry groups that contain only rotations are called cyclic and those that contain reflections are called dihedral.

Periodic Patterns – Infinite Symmetry Groups

The second type of pattern contains translational symmetries in two independent directions (Figure 7). These patterns are classified by the smallest angle of rotation that appears in the pattern. By a property known as the crystallographic restriction, the only angles that may appear in rotations for these patterns are 60°, 90°, 120°, 180°, or the pattern may have no rotational symmetry. If each of these patterns are considered to continue repeating in all directions then their symmetry groups are infinite. It can be shown that there are exactly seventeen symmetry groups for these types of patterns (Figure 8) and they are referred to as the crystallographic or wallpaper groups.

Geometric Constructions

Appealing to the visual intuition of students, Islamic tilings may be used to introduce topics from geometry in a way that may replace or complement the traditional axiomatic development of the subject. Projects, which challenge the students to construct tilings, may be designed to lead students through a desired series of concepts. In addition, if the students are given freedom to

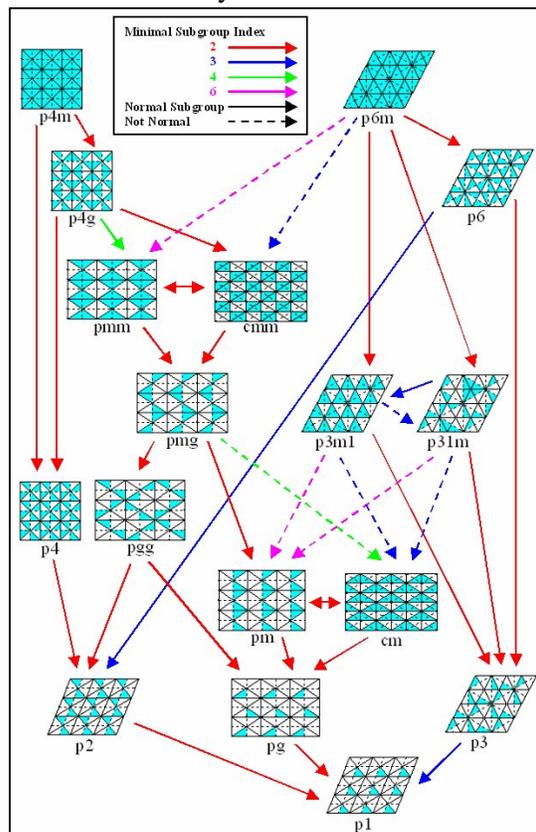


Figure 8 The Seventeen Crystallographic Groups for Periodic Tilings

make their own design decisions, this may provide the class a direction with a life of its own. By way of clarification, the following example, which is appropriate for a class in modern geometry, abstract algebra, or general education, is given. The exercise includes the original construction



Figure 9 Tiling with 8-Point Stars from the Alhambra

problem that is to be posed to the class along with a potential sequence of topics through which the class may be directed in solving the problem. This collaborative problem is designed for use with a wide range of student majors who may or may not be familiar with concepts in group theory.

The Construction Problem – The class is given a graphic of an Islamic tiling from the Alhambra (Figure 9) and asked to develop a strategy for constructing a pattern similar to it. The students work in and outside of class over the course of two or three weeks on their own construction of the pattern. Each class meeting is involved with a discussion of discoveries made by students as well as challenges they might encounter. The following list contains a sequence of questions that are addressed by the problem along with the corresponding topic that is covered. Each student is allowed to use a straightedge and compass as well as Geometer’s Sketchpad.

Questions for the class (Topic covered by the question)

- How should a construction of the tiling begin? Is there a particular tile that may produce the overall tiling? (Tiling by translation)
- Does it matter if a tile with a star, a tile with a cross, or some other tile is chosen? (Independence of generating tile)
- If a star is chosen as the focus, an interesting side question would be; Why do certain patterns contain stars with a particular number of points? Obviously, this type of question may lead in a variety of directions involving culture, aesthetics, history, or design considerations. (Open-ended question)
- Can overlapping be added to enhance the pattern? Note that for this example, the overlapping that was chosen alters the symmetry from the original Alhambra tiling. (Symmetry breaking)
- What are the symmetries of the tile? (Symmetry groups)
- What is the minimal tile that is needed to be able to reproduce the entire pattern? (Fundamental region for the tiling)
- How does the introduction of color affect the symmetry of the pattern? (Subgroups)
- Does your tiling contain the same symmetries as the original tiling? Can you find a different tiling from the Alhambra that contains the same symmetries as your tiling? (Isomorphism)

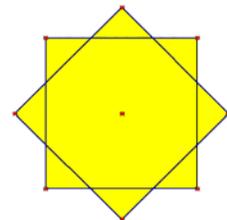


Figure 10 Eight-Point Star

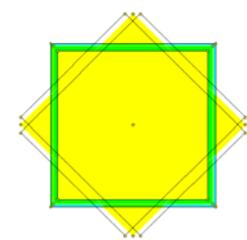


Figure 11 Star Lattice Added

A sequence of constructions drawn with Geometer’s Sketchpad is shown (Figures 10 – 13) that reproduce the essence of the given Alhambra tiling.

Since the progress on these questions relies heavily on student input, there are both positive and negative aspects to this type of approach. On the one hand, it can be a rewarding experience if students get excited by the challenge of developing their own individual construction and they set an energetic pace for the class. At its best, this technique can allow for students to teach one another as they learn from their own discovery process. On the other hand, there is potential for the class to get bogged down on a particular idea or to

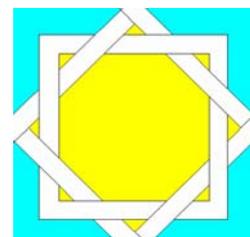


Figure 12 Lattice Overlap Added

take the discussion in a direction that runs contrary to what the instructor had originally planned. In the event, these negative scenarios begin to arise; the instructor may revert to a more structured lesson plan that would achieve many of the desired learning objectives.

Conclusion – Comparing and Contrasting the Past with the Present

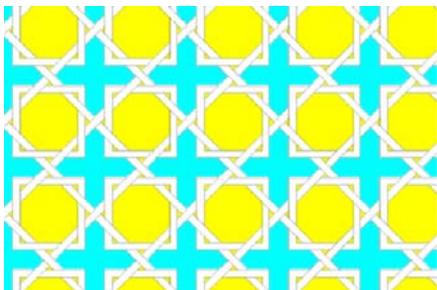


Figure 13 Tiling Constructed with Geometer's Sketchpad

Educators are continually revising classroom strategies for making the study of mathematics more interesting and meaningful for their students. By introducing history, culture, and art into discussions in the classroom, students have an opportunity to see mathematics as a multidimensional embroidery woven across time and space.

One can only imagine how magnificent the Islamic art and ornamentation of the Alhambra must have been in the 14th or 15th Century. It is indeed a stimulating exercise for students to attempt to understand the beautiful symmetry of these Islamic tilings and learn the geometry required to reconstruct them.

Also, with the aid of modern tools like Geometer's Sketchpad, students may gain further appreciation to what was accomplished by artists and geometers of bygone centuries who possessed the simplest of tools.

While touring Granada this summer and viewing the marvelous Islamic art and architecture of the Alhambra Palace, I had the opportunity to see firsthand a historical connection of the past to the present. From the grounds of the Alhambra, I could look out over the Rio Darro to see a new mosque that had just been completed on the nearby hillside. Two weeks earlier, His Highness Dr. Sheikh Sultan bin Mohammed Al Qasimi, Member of the Supreme Council of the United Arab Emirates and Ruler of Sharjah, had visited Granada to inaugurate the Granada Mosque (Figure 14) and its cultural center completing a historical connection across five centuries.



Figure 14 The Granada Mosque in Granada, Spain

References

- Acedo, A. (2000). *The Alhambra and Generalife in Focus*. Edilux S. L.
- National Council of Teachers of Mathematics. (1995). *Principles and Standards for School Mathematics*. Retrieved July 10, 2003 from <http://standards.nctm.org/>.
- Saliba, G. (1999). *Whose science is Arabic science in Renaissance Europe?* Retrieved March 15, 2002 from <http://www.columbia.edu/~gas1/project/visions/case1/sci.1.html>.
- Tennant, R. (2002). *Subgroup Lattices for Crystallographic Groups*. VISMATH Electronic Journal vol. 4 no. 4, Retrieved August 15, 2003 from <http://members.tripod.com/vismath8/tennant/index.html>.
- Tennant, R. (2003). *Islamic Constructions: The Geometry Needed By Craftsmen*, BRIDGES/ISAMA International Conference Proceedings. 459 - 463.
- Weyl, H. (1952). *Symmetry*. Princeton University Press.