Magic Geometry: Mosaics in the Alhambra

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Abstract

There is much to see and admire in the Alhambra Palace: exquisite rooms, decorated with stone and wood carvings, finest ornaments, and calligraphy; night sky represented in ceilings built of thousands of pieces of wood; gardens, courtyards and fountains; monuments, towers, archways - the list is endless. Quite possibly, an immense wealth of ornamental patterns, friezes, mosaics, star designs, and brickwork motifs tops the list. Among those, mosaics are perhaps the most interesting and the most intriguing. Scientists and artists working in the Islamic world pushed geometry to its limits, creating patterns and configurations whose sophistication has never been surpassed. Investigating numerous possibilities, based on experience and long tradition, builders of mosaics in the Alhambra created them all – in the sense of the mathematics theorem on the classification of plane crystallographic groups (or wallpaper patterns). Mosaics are an invitation for a 'dynamic' experience, different from enjoying a picture of a landscape. Our eye is not able to focus on one location; there is no centre, no boundary and no preferred direction. This paper suggests the use of a vocabulary from geometry to express some of our visual experiences related to mosaics.

Introduction

'Philosophy is written in this grand book, the universe, which stands continually open to our gaze. But the book cannot be understood unless one first learns to comprehend the language and read the letters in which it is composed. It is written in the language of mathematics, and its characters are triangles, circles and other geometric figures without which it is humanly impossible to understand a word of it; without these one wanders about in a dark labyrinth.'

Art is not philosophy, philosophy is not art. Nevertheless, the famous Galileo's message can easily be modified to include the domain of the artistic. To formulate thoughts and ideas about what we see, to express our visual experiences, or to communicate visual information, we use a language. Although not the language of mathematics, this language has been enriched with a number of mathematics 'characters' - proportion, symmetry, curvature, dimension, perspective, and space, to name a few.

Through a study of mosaics in the Alhambra Palace (Granada, Andalusia, Spain), this essay will illustrate several important mathematical 'characters' that have found their way into the world of art. Although appropriate to describe various geometric features of mosaics in particular, and art in general, the concepts of rigid motions and symmetry and other geometric ideas extend, within mathematics and some other disciplines, far beyond the domain of the geometric (i.e., the visual).

What is a Mosaic?

The term 'mosaic' usually refers to a decorative/art form where the image (two-dimensional, flat or curved) is constructed using small coloured pieces of glass, stone, tile, ceramics, or some other material. In the context of this exposition, we adopt a somewhat narrow, but more appropriate meaning. We think of a mosaic as a decorative or art form where a certain recognizable pattern (sometimes called a matrix) is repeated in two dimensions (see Figure 1). The important part of the definition is the requirement that the iterations of the pattern occur in *two* dimensions i.e., in two distinct directions, such as horizontal and vertical.

Formalizing this intuitive idea of a mosaic, we obtain the corresponding mathematical concept: a tiling. To form a tiling means to cover a plane with various geometric forms (such as polygons or shapes bounded by curves – so-called 'tiles') in a way that leaves no gaps (the whole plane is covered) and does not allow for overlaps (the tiles cannot be partially of fully placed on top of each other). Sometimes the word 'tessellation' is used in lieu of 'tiling.'

We think of mosaics in the way they were meant to be viewed and thought about - as infinite extensions of the pattern that is actually shown. Being exposed to a small part, we contemplate the whole. The word 'plane' in the mathematical definition represents that two-dimensional, unbounded surface where the mosaic extends ('unfolds').

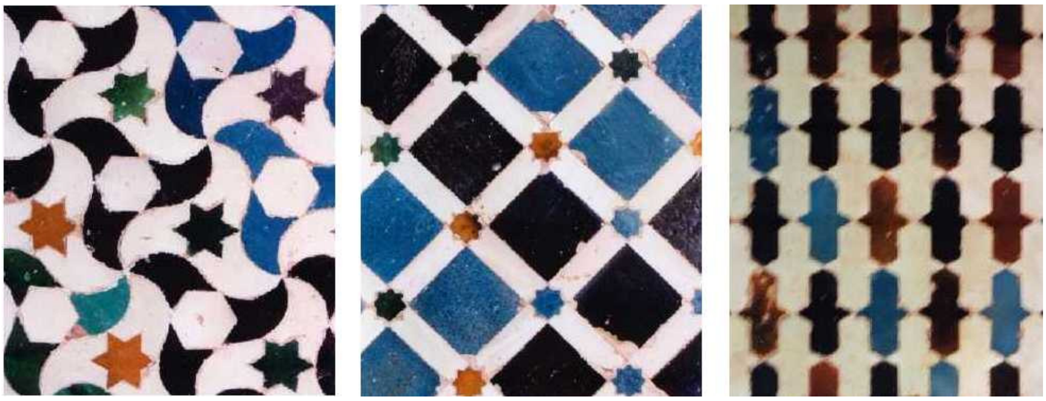

Figure 1: Three mosaics from the Alhambra Palace.

Geometry in Islamic World

Although mosaics can de identified among the artifacts created by people in almost every culture, past or present, nobody has designed and used them with the skill, knowledge and sophistication of the artisans and artists working in the Islamic world[1]. The key to our understanding of this phenomenon lies, perhaps not surprisingly, in geometry.

There are several reasons why geometry played an important role in the Islamic world. First of all, geometric forms, patterns and configurations provided a desired level of abstraction. Islam being a

[1] Mostly between 10^{th} and 16^{th} c AD.

non-figurative religion, its theologians selected geometry as a medium of conveying images of God.

Secondly, geometry was viewed as a force capable of unifying the domains of the secular and the religious: ‘the study of sensible geometry leads to skill in all practical arts, while the study of intelligible geometry leads to skill in intellectual arts because this science is one of the gates through which we move to the knowledge of the essence of the soul, and that is the root of all knowledge.’[2]

Finally, intricate knowledge of properties of geometric objects (such as circles and lines) and sophisticated computational skills became essential for the development of astronomy. Accurate astronomical observations were employed in the production of almanacs and calendars, for time-keeping, and as a tool of navigation (at sea, and on land, especially in large deserts).

Geometric patterns can be found virtually everywhere: in tile mosaics, friezes and other wall ornaments, arabesques, coffered ceilings, floors, carpets, manuscripts, wooden carvings, doors, screens, executed in a variety of media (tiles, bricks, marble, wood, brass, paper, plaster, glass, etc.).

Symmetry – Mathematics Background

In order to give a precise meaning to the word ‘symmetry,’ we need to introduce the geometric concept of a rigid motion. A rigid motion of an object is a motion that keeps unchanged the dimensions (i.e., size, shape, angles, distances) and relative position of the points on that object. Throwing a ball (but not breaking it or deforming it in any way), or driving a car (assume ideal situation, i.e. no accidents) are examples of rigid motions. There are four types of rigid motions in the plane: translation, reflection, rotation and glide reflection.

Translation is a motion of an object along a line in the given direction for a given distance, in such a way that what is horizontal remains horizontal, and what is vertical remains vertical throughout the motion (i.e., there are no turns). A reflection is determined by a line (called a mirror line, a line of reflection, or an axis of reflection). Placing an object on one side of the mirror line creates the image on the other side. For instance, a reflection (with respect to a vertical line) transforms the symbol < into the symbol >.

A rotation is a turn around the given point (called centre of rotation) for a given angle (called angle of rotation). According to a convention, all rotations are performed counterclockwise. A glide reflection is a combination: first, a mirror reflection, and then a translation in the direction parallel to the mirror are performed. Our footprints in snow - assuming we are walking along a straight line - are glide reflections of each other: the two consecutive footprints are mirror images of each other, but are located at a distance (length of our step) apart from each other.

We are now ready to give a precise definition of what it means to be symmetric.

Any rigid motion that leaves the given object apparently unchanged (‘invariant’ in mathematics language) is called a symmetry of that object. For instance, placing a mirror along a diagonal of the square will keep the square’s appearance unchanged (Figure 2, left). The remaining mirror reflections that are symmetries of the square are shown in dashed lines in Figure 2 (middle).

[2] From the treatise *Rasai’il*, by scholars belonging to the Brotherhood of Purity (Ikhwan Al-Safa), 9th -10th c AD.

Rotations of the square for 0 degrees (don't rotate at all), 90 degrees (quarter-turn), 180 degrees (half-turn), or 270 degrees (three-quarters-turn) around its centre will keep it invariant. However, the rotation about its centre for 45 degrees (one-eight of the turn; Figure 2, right) does not keep the square's appearance unchanged, so it is not a symmetry of the square.

In other words, mirror reflections (along the diagonals and parallel to the sides, going through the centre of the square) and rotations (about its centre, for 0, 90, 180, and 270 degrees) are symmetries of the square.

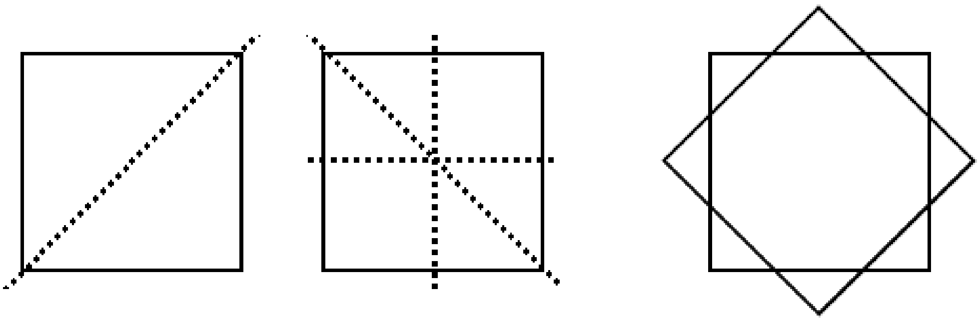

Figure 2: Square and its symmetries.

A Case Study

As an illustration of the concept of symmetry, we consider the mosaic in Figure 3 (left), found in a number of locations in Alhambra[3]. The exact motif, or slight modifications of it, can be located in numerous places throughout the (former and present) Islamic world.

In Figure 3 (centre and right) we show two different translations that are symmetries of the mosaic. Arrows indicate direction and length of the translation. As usual, we assume that the mosaic extends in all directions to infinity. (Note than translations are always present, since a mosaic is constructed by repeating a pattern.)

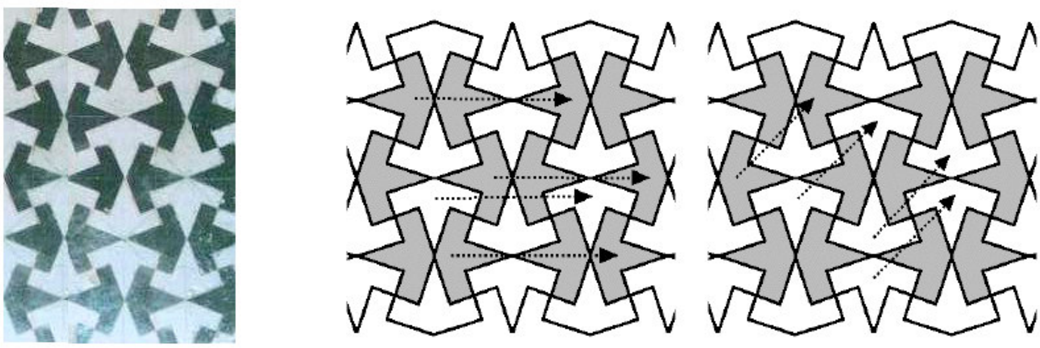

Figure 3: Mosaic and its translation symmetries.

[3] For instance, in niches of the entrance arches to several rooms.

The mosaic under consideration has a number of more interesting symmetries: mirror reflections are shown in Figure 4 (left), the dashed lines indicating the mirrors.

There are two types of rotations that keep the mosaic's appearance unchanged: half-turn (180 degrees) rotations centred at A, and quarter-turn (90 degrees) rotations centred at B (Figure 4, centre). Note that the half-turn rotation respects the colouring (i.e., white tiles get moved to the white tiles, black tiles to black tiles), whereas a quarter-turn moves white tiles to black tiles, and vice versa.

Finally, there is also a glide reflection (Figure 4, right). The line of reflection (labeled g) is located half-way between the vertical mirrors (labeled m). The length of the translation is equal to the height (or the width, they are equal) of the given shape. The tiles with the same letter correspond to each other under the glide reflection.

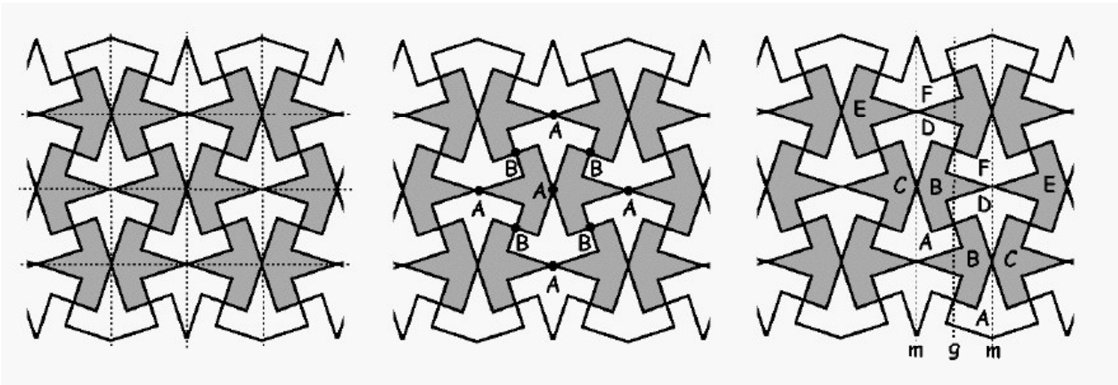

Figure 4: Symmetries of the mosaic in Figure 3.

In conclusion, the mosaic in Figure 3 (left) possesses a number of symmetries: translations, two types of mirror reflections (with respect to horizontally and vertically placed mirrors), two types of rotations (half-turn and quarter-turn) and a glide reflection.

Plane Crystallographic Groups and Classification of Mosaics

Imagine that we take all mosaics that we can find in the Alhambra (or all mosaics in the world!), list their symmetries (as we did in the case study in the previous section), and throw mosaics that possess the same symmetries into the same box.

For instance, the mosaics in Figure 5 (left and centre) belong to the same box as the mosaic we just analyzed (in Figures 3 and 4). However, the mosaic on the right in Figure 5 has additional symmetries (diagonal mirrors), and therefore needs to be placed into a different box.

Note that the three mosaics in Figure 1 belong to different boxes (think of them as two-colour mosaics: white remains white, all other colours are viewed as one colour, say black[4]). The mosaic on the left has a third-turn (360/3=120 degrees) rotational symmetry, that neither of the other two has. The one in the centre has more reflections than the other two (neither of the remaining two has a diagonal mirror symmetry). The mosaic on the right has a half-turn rotational symmetry, but has

[4] If we take full colour into consideration, all symmetries are lost.

neither the quarter-turn rotational symmetry (as the one in the middle), nor the third-turn rotational symmetry (as the one on the left).

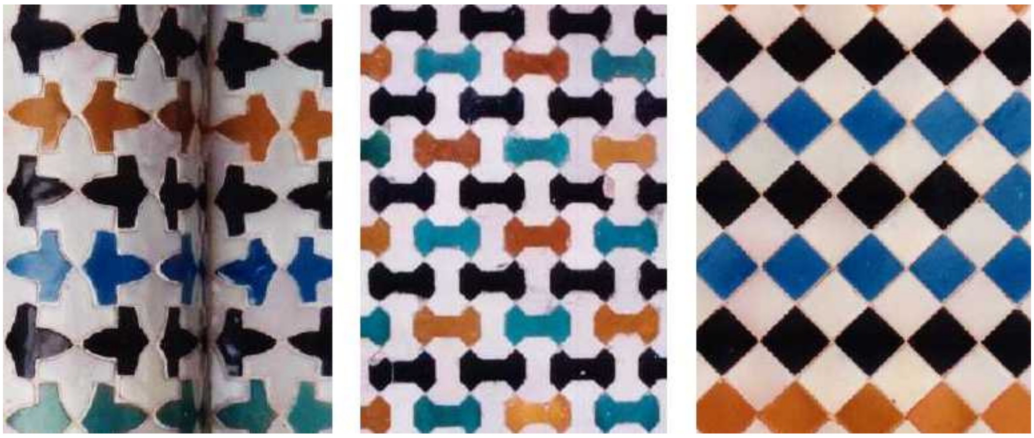

Figure 5: Mosaics from the Alhambra Palace.

We might wonder: how many different boxes are there?

A branch of mathematics, called group theory, provided the answer to our inquiry. Mathematicians have studied objects called 'groups' in connection with various problems, one of which is the investigation of properties of crystals. The fact that there are 17 different 'plane crystallographic groups' was first proved in 1891[5]. Later, the statement of the theorem was reformulated as the fact that 'there are seventeen different wallpaper designs,' echoing the study in the context of two-dimensional tilings. Thus the answer to our question – there are seventeen different boxes!

Alhambra and Seventeen Wallpaper Designs

Thus, we can identify any given mosaic as belonging to one of the seventeen groups. The existence of a classification algorithm (presented in the next section) makes this task more-or-less easy to do.

It is not known who, or when, first claimed that representative mosaics belonging to *each* of the seventeen groups exist in the Alhambra. The claim has never been a topic of intense research in mathematics, mostly because it was not a question of interest to mathematics theory. Nevertheless, it has quite an interesting history. In a thesis published in 1944, the author demonstrated that the claim was false [Mü]. In subsequent decades, several papers were published, identifying some wallpaper designs in the Palace - but never all 17 of them. Finally, in mid-1980s, it was verified that the claim was true [GGS, PG]. In other words, all seventeen plane groups (wallpaper designs) are present among the mosaics in the Alhambra.

Most certainly, Alhambra was not built as a 'catalogue' of plane symmetric groups. Hundreds of years of experience in constructing ornaments and various other decorations, and involvement of

[5] Proved by a E.S. Fedorov, and published in Russian. The proof became known after G. Polya published a paper '*Uber die Analogie der Kristallsymmetrie in der Ebene*' in 1924.

many thousands of geometers, artists and artisans throughout the Islamic world created a richness of designs that happened to exhaust all theoretical possibilities.

Classification of Wallpaper Designs

The algorithm[6] used for classification (see Figure 7) is based on analysis of symmetries of the pattern. To start, we identify the smallest rotation symmetry (smallest in terms of the angle of rotation). It can be proven that the only possibilities for the rotations are half-turn, third-turn, quarter-turn and sixth-turn (i.e., rotations by 180, 120, 90 and 60 degrees respectively)[7]. It is possible that the design has no rotation symmetry, other than the trivial, full-turn (360 degrees) rotation (called ‘none’ in Figure 7). The symbols identifying the designs (cm, pm, pg, p1, etc.) belong to the so-called IUC notation[8].

The case study mosaic (Figures 3 and 4) belongs to the group pmm. If we look at the underlying uncoloured pattern (i.e., view all tiles as white), then the mosaic belongs to the p4g group. Considering the underlying uncoloured patterns, the mosaics in Figure 1 are classified as (from left to right) p3, p4m and pmm. The same classification is obtained if we include colouring in the following way: white remains white, all other colours are viewed as being the same colour. If we decide to consider the colours as they are, then all symmetries are lost, and the three mosaics fall into the p1 group.

The underlying uncoloured pattern in Figure 5 (centre) is p4g. Considering the pattern formed by the black tiles (Figure 6, left), we obtain a pmm design. Pattern formed by the blue tiles (Figure 6, right) is a cmm design. Pattern formed by the yellow (honeyed hue) tiles is the same as the one formed by the blue tiles. The dot labeled C in Figure 6 (right) indicates the centre of the half-turn rotation which does not lie on the line (axis) of reflection. That is what makes it a cmm design and not a pmm design. The mosaic in Figure 5 (centre) viewed exactly as it is (i.e., considering symmetries that preserve all colours), is a pmg design.

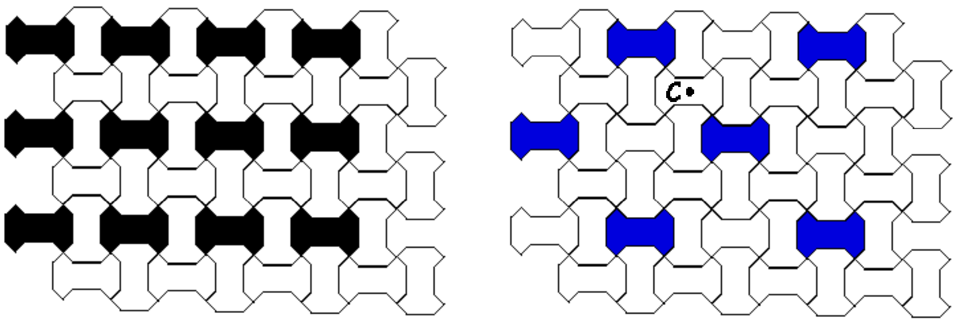

Figure 6: Pattern from Figure 5 (centre) viewed as coloured in two different ways.

[6] From the website ‘Symmetries of Culture,’ http://members.tripod.com/vismath6/crowe1/ by Donald W. Crowe, Department of Mathematics, University of Wisconsin-Madison.
[7] This fact is known as the *crystallographic restriction*.
[8] International Union of Crystallography notation for symmetry groups (adopted in 1952).

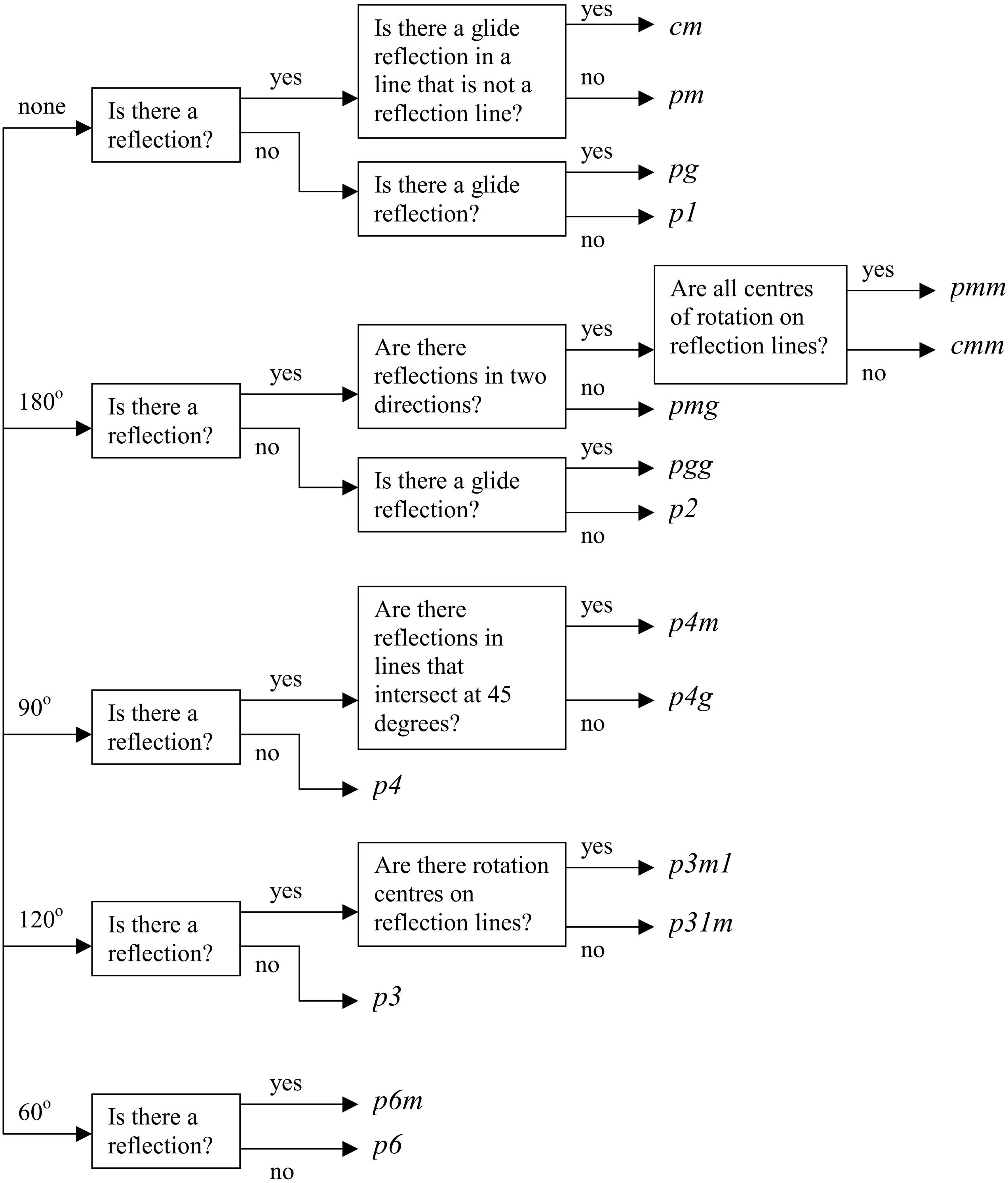

Figure 7: Classification of wallpaper designs.

Closing Remarks

In this paper we presented ceramic tile mosaics only. However, in order to justify the claim that all seventeen wallpaper designs are present in the Alhambra, we have to include mosaics in stucco wall decorations (such as the one if Figure 8, left), frescoes, ceiling and floor motifs, and also in miniature decorations within ornaments (for example in muqarnas[9]).

There are other issues related to the validity of the claim. Several mosaics have been removed from the Palace and placed into museums[10], some mosaics have been damaged to the extent that it is impossible to classify them with a high degree of accuracy, and an unknown number of mosaics have disappeared in successive reconstructions of the Palace[11].

The classification of mosaics depends on several artificial assumptions, for instance about colouring or interlacing. Examples discussed in the previous section show how classification changes depending on the way we look at colouring. Furthermore, inscriptions and ornamentation are usually ignored, since they destroy most of the symmetries.

There is no agreement on how to view interlacing or layers (i.e., how to treat over- and under-crossings). Consider the mosaic in Figure 8 (left) and the elliptic pieces labeled A and B (Figure 8, centre and right): piece A is taken from the upper left part of the mosaic, and piece B can be located slightly below the centre. Piece A has no reflections, whereas piece B has a vertical mirror symmetry[12]. Thus, considering interlacing as is, the mosaic in Figure 8 (left) is p1; if we ignore interlacing (i.e., imagine that all elliptic pieces look like the piece B), the mosaic belongs to the group cm.

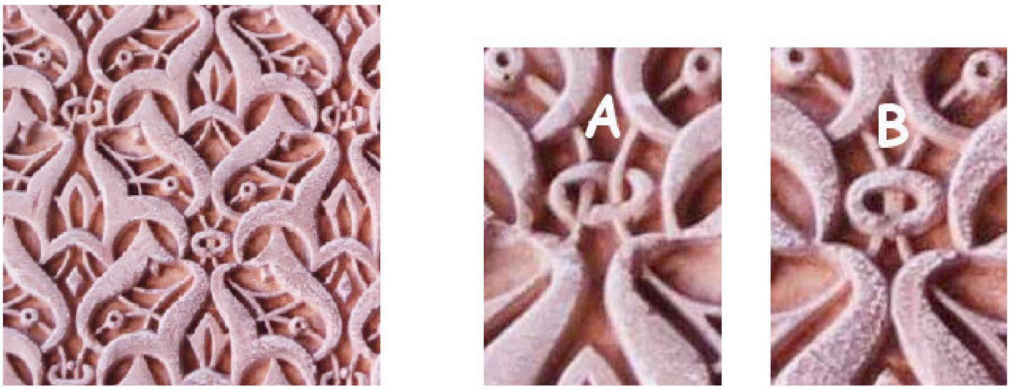

Figure 8: Mosaic in a stucco wall ornament.

[9] Hanging stalactite ornaments in rooms and arches.
[10] Museum of the Alhambra, for instance. The author has learnt that some mosaics were moved to other museums in the region of Andalusia.
[11] The author was a witness to a repair of a part of the floor in the Palace that completely disregarded the existing patterns.
[12] It has horizontal mirror symmetry as well, but it does not count (since the whole mosaic does not have it).

Thus, the best answer to the claim is the following. With an appropriate set of conventions (on colouring, interlacing, ornamentation, etc., that are too technical and cumbersome to be quoted here), all seventeen wallpaper designs are present in the Alhambra (see [GGS] and [PG] for details).

Upon arrival to the Alhambra, the author realized that locating representatives of all wallpaper designs became secondary to the excitement of investigation and sheer enjoyment of beauty of ornamentation and architecture.

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